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ROBINSON FORCING IS NOT ABSOLUTE†

BY
LARRY MICHAEL MANEVITZ

ABSTRACT
Robinson (or infinite model theoretic) forcing is studied in the context of set theory. The major result is that infinite forcing, genericity, and related notions are not absolute relative to ZFC. This answers a question of G. Sacks and provides a non-trivial example of a non-absolute notion of model theory. This non-absoluteness phenomenon is shown to be intrinsic to the concept of infinite forcing in the sense that any ZFC-definable set theory, relative to which forcing is absolute, has the flavor of asserting self-inconsistency. More precisely: If T is a ZFC-definable set theory such that the existence of a standard model of T is consistent with T, then forcing is not absolute relative to T. For example, if it is consistent that ZFC + "there is a measureable cardinal" has a standard model then forcing is not absolute relative to ZFC + "there is a measureable cardinal." Some consequences: 1) The resultants for infinite forcing may not be chosen "effectively" in general. This answers a question of A. Robinson. 2) If ZFC is consistent then it is consistent that the class of constructible division rings is disjoint from the class of generic division rings. 3) If ZFC is consistent then the generics may not be axiomatized by a single sentence of $L_{\omega_1\omega}$.

0. Introduction

Abraham Robinson introduced forcing to model theory in a series of papers in 1969–70 ([37], [38], [39]). Motivated both by Paul Cohen's work in set theory and by the idea of "potential satisfaction of existential formulas" [35], he created two distinctive types of forcing: finite forcing (which closely resembles Cohen forcing) and infinite forcing.

Finite forcing has proved a useful tool in model theory ([26], [45]); it turns out to be a way to avoid using the Omitting Types Theorem.

Infinite (or "Robinson") forcing, on the other hand, has not produced dramatic applications. Its main virtue, so far, seems to be an added insight into

† In Memoriam: Abraham Robinson

the notion of "algebraic closedness" [40]. (Infinite forcing provides a canonical way of constructing an algebraically closed extension of any model of an inductive theory. These models (generics) may then be studied for their structure theory, etc., independently of the forcing definition. The added insight gives a remarkably short proof of Lindström's theorem that an inductive $\aleph_1$-categorical theory with no finite models is model complete.)

This paper gives one reason for the lack of applicability of infinite forcing — it is not absolute relative to ZFC.

In fact, it must be non-absolute relative to any "true" set theory. (This answers a question of G. Sacks [43]). This is pertinent since, as Sacks has pointed out [43], most model-theoretic concepts are absolute.

The outline of this paper is as follows: §1 contains the basic definitions and results of forcing and set theory pre-requisite for this work. §2 contains the proof that Robinson-forcing is not absolute relative to ZFC. Consequences are also included. Among these are: 1) Generics may not be axiomatized by a single sentence of $L_{\omega_1\omega}$ (previously proved by A. Macintyre). 2) If ZFC is consistent then it is consistent that the class of generic division rings formed in Gödel's constructible universe is disjoint from the class of generic division rings. 3) The resultants for infinite forcing may not be chosen "effectively" under any sensible notion of "effectiveness." This answers a question of Robinson.

A discussion of the absoluteness of the levels of the approximating generic hierarchies is also included.

§3 points out that these results are not a consequence of the particular choice and strength of ZFC. In fact, no "true" extensions of ZFC will be strong enough to require that Robinson-forcing be absolute relative to it.

More precisely: If ZFC + Φ + "ZFC + Φ has a standard model" is consistent, then infinite forcing and related concepts are not absolute relative to ZFC + Φ. For example, take Φ to be the single sentence asserting the existence of a measurable cardinal, or take Φ to be $V = L$.

It is also pointed out that the assumption $(T^F)^{(L)} = T^F$ for all countable constructible T is equivalent to $P(\omega) = P(\omega)^{(L)}$. This answers a question raised by M. Boffa and A. Macintyre.

This work consists of a major part of the author's dissertation, Yale, 1975. The problem was suggested by A. Robinson in the fall of 1972, and about half of the work was done under his supervision. The remainder was completed while A. Macintyre served as thesis advisor.

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1. Preliminaries and background

I. *Forcing*

The basic reference for infinite model theoretic forcing is [37]. A very readable and comprehensive account of the development of the theory appears in [19] and the reader is referred there for further reference. Accordingly, it will suffice here to state the bare essentials that are required for this work. Most proofs are omitted; they may be found in [19]. The model theoretic notation is standard; the encyclopedic reference is [7].

The meta-theory for this paper is ZFC.

1.1. Let M be a structure for a language L. The language of M, denoted $L(M)$, is the expansion of L to a language including constants for all the members of M. (The precise method of assigning constants is irrelevant as long as the constants are new symbols. It will be assumed that if $M \subset N$ then $L(N)$ is an expansion of $L(M)$ in the obvious way. This point will not be further belabored).

1.2. Definition. Let T be a consistent theory in some language. Let $\Sigma(T)$ be the class of all models of T. The relation M *(infinitely Robinson) forces* φ, $M \Vdash_T \varphi$, between elements of $\Sigma(T)$ and sentences in their language is defined by induction:

i) If φ is atomic then $M \Vdash_T \varphi$ precisely when φ holds in M.

ii) If φ is of the form $\psi \wedge \chi$ then $M \Vdash_T \varphi$ precisely when $M \Vdash_T \psi$ and $M \Vdash_T \chi$.

iii) If φ is of the form $\exists x \psi(x)$ then $M \Vdash_T \varphi$ precisely when there is a constant a, in the language of M, such that $M \Vdash_T \psi(a)$.

iv) If φ is of the form $\sim \psi$ then $M \Vdash_T \varphi$ precisely when there is no extension N of M in $\Sigma(T)$ such that $N \Vdash_T \psi$.

Robinson's original setting is more general than that presented here (a fact exploited by G. Cherlin [8] in particular). However, this suffices for the present investigation.

Robinson proved the "usual" forcing lemma:

1.3. PROPOSITION. *For any theory T, for any $M \in \Sigma(T)$ and for any $\varphi \in L(M)$, either $M \Vdash_T \varphi$ or $M \Vdash_T \sim \varphi$.*

1.4. PROPOSITION. *If M and N are in $\Sigma(T)$ with $M \subseteq N$ and $\varphi \in L(M)$ then $M \Vdash_T \varphi$ implies $N \Vdash_T \varphi$.*

The proof of 1.4 is quite typical of elementary results in forcing theory; it proceeds by induction following Definition 1.2. *But what is the relation over which this induction is performed?*

The relation implicit in 1.2 is not well-founded. (This is a result of clause iv.) Thus, this definition is not, as it stands, realizable in ZFC. Of course, there is the expectation that an appropriate reformulation will capture the forcing concept in ZFC. This expectation is justified as can be seen from 1.6. However, that this is not automatic may be seen from the following example.

1.5. EXAMPLE. M. Boffa [5] introduced a modification of Robinson forcing designed to deal only with structures with a binary relation, E. He then alters clause (iv) of Robinson's definition replacing "extension" by "end extension" (recall $\langle N, F\rangle$ is an end extension of $\langle M, E\rangle$ if $\langle N, F\rangle$ is an extension of $\langle M, E\rangle$ and if $a \in L(M)$ and $N \models bFa$ then $b \in L(M)$). The results in [5] (e.g. that a Boffa-generic satisfies ZFC except for the axiom of regularity) shows that the consistency of ZFC is provable whenever his motion of forcing is definable. Thus Boffa-forcing is *not* definable in ZFC. The difference between Robinson and Boffa forcing is due mainly to a failure of the analogue for Boffa's theory of 1.6 below.

1.6. PROPOSITION. (Löwenheim–Skolem Theorem for Forcing). *Let λ be the maximum of the cardinalities of T and $\aleph_0$. If $M \in \Sigma(T)$ and $M \Vdash_T \varphi$ then there exists an $M' \in \Sigma(T)$ such that M' has cardinality λ, $M' \subseteq M$ and $M' \Vdash_T \varphi$.*

Thus, it is clear that the same concept results if a cardinality bound is put into clause (iv) of 1.2. This modified definition may then be presented formally inside ZFC. This process in both laborious and straightforward and will be omitted here. The rigorous verification that the appropriate notion is captured relies on Beth's theorem. (Full details appear in [31].)

To sum up, it may now be assumed that the relation $M \Vdash_T \varphi$ is expressible in

ZFC. In fact the meta-theory for both general forcing theory and the remainder of this work may henceforth be assumed to be ZFC.

1.7. DEFINITION. An element M of $\Sigma(T)$ is called T-generic (or Robinson infinite generic for T) if for any sentence φ in the language of M, either $M \Vdash_T \varphi$ or $M \Vdash_T \sim \varphi$. (This is equivalent to the condition $M \Vdash_T \varphi$ if and only if $M \vDash \varphi$.)

1.8. DEFINITION. A theory T is inductive if its class of models is closed under the union of chains. (By the Los-Suszko Theorem [36, 3.4.7] this means T is equivalent to a theory axiomatized by $\forall\exists$ sentences.)

1.9. PROPOSITION. *If T is inductive then every model of T is contained in a T-generic model.*

1.10. DEFINITION. Let T be inductive. T^F is the set of sentences that hold in all T-generic models. This implies $T^F = \{\varphi \mid M \Vdash_T \sim \varphi$ for all $M \in \Sigma(T)\}$.

A. Macintyre [28] and W. Wheeler [49] investigated the (recursive-theoretic) complexity of T^F when T is an axiomatization of division rings. They discovered (using different methods) that in this case T^F is recursively equivalent to full second order number theory.

D. Goldrei, A. Macintyre and H. Simmons [16] and J. Hirschfeld [17] noticed the same phenomenon when T is full first order number theory, i.e. T^F is recursively equivalent to full second order number theory. Macintyre [27] (and later Wheeler [49]) pointed out that this also occurs when T is an axiomatization of group theory.

1.11. PROPOSITION. *Let T be axiomatization of group theory in the language $(\cdot, e)$. Then T^F is recursively equivalent to the true sentences of second order arithmetic.*

1.12. PROPOSITION. *Let T be an axiomatization of division rings in the language $(+, \cdot, 0, 1)$. Then T^F is recursively equivalent to the true sentences of second order arithmetic.*

1.13. PROPOSITION. *Let T be the set of all true sentences in arithmetic in the language $(+, \cdot, 0, 1)$. Then T^F is recursively equivalent to the true sentences of second order arithmetic.*

The equivalences in 1.11, 1.12 and 1.13 may be given by a one-to-one recursive function, f, such that $f(\sim \psi) = \sim f(\psi)$.

II. *Set Theory*

Knowledge of sophisticated set-theoretic machinery is not a prerequisite for this work. For example, there are no Cohen forcing arguments. On the other hand, several deep results are needed at various points. Basic requirements are summarized here. For background reference see Fraenkel, Bar-Hillel and Lévy [14]. Other useful books are Cohen [9], Jech [21], Drake [11] and Devlin [10].

1.14. ZF(C) is the usual set of axioms of Zermelo-Frankel Set Theory (with the Axiom of Choice) formulated in the language with one binary relation, $\in$. The axioms of ZF are Extensionality, Pairing, Union, Power-Set, Infinity, and the schemes of Separation, Replacement and Regularity.

1.15. A set z is transitive if $x \in y \in z$ implies $x \in z$. $TC(x)$ is the smallest (by inclusion) transitive set containing x. $|x|$ is the cardinality of x. $H(\lambda) = \{x \mid |TC(x)| < \lambda\}$ where λ is a cardinal.

1.16. *A bounded quantifier* is one of the form $\forall x\ (x \in y \rightarrow \varphi)$ or of the type $\exists x\ (x \in y \wedge \varphi)$. Other quantifiers are *unbounded.* A formula is $\Sigma_0 = \Pi_0$ if it contains no unbounded quantifiers. For $n \geqq 1$, φ is Σ_n if it is of the form $\exists x \psi(x)$ where ψ is Π_{n-1}. φ is Π_n if it is of the form $\forall x \psi(x)$ where ψ is Σ_{n-1}.

If T is a theory in the language of $\in$, a formula, φ is $\Sigma^T_n(\Pi^T_n)$ if there is a formula, ψ, such that $T \vdash \varphi \leftrightarrow \psi$ where ψ is Σ_n (resp. Π_n). A formula is Δ^T_n if it is both Σ^T_n and Π^T_n.

1.17. M is a *class* if M is the collection of elements, x, such that $\varphi(x)$ holds, where φ is a formula in the language of $\in$.

If E is a binary relation on M and φ is a sentence in the language of $\in$, then $\varphi^{(M)}$ or $\varphi^{(\langle M,E\rangle)}$ means $\langle M. E\rangle \models \varphi$.

If $\langle M, E\rangle$ and $\langle N, F\rangle$ are structures with binary relations, then $\langle N, F\rangle$ is said to be an *end-extension* of $\langle M, E\rangle$, $\langle N, F\rangle \supset_{\text{end}} \langle M, E\rangle$, if $\langle N, F\rangle \supset \langle M, E\rangle$ as structures and whenever $m \in M$, $n \in N$ and nFm then $n \in M$.

A formula $\varphi(x_1, \cdots, x_n)$ in the language of $\in$ is said to be *absolute relative to a theory* T (in the language of $\in$) if whenever $\langle M, E\rangle \models T$, $\langle N, F\rangle \models T$, $\langle M, E\rangle \subset_{\text{end}} \langle N, F\rangle$, and $\{m_1, \cdots m_n\} \subset M$ then $\langle M, E\rangle \models \varphi(m_1, \cdots m_n)$ if and only if $\langle N, F\rangle \models \varphi(m_1, \cdots, m_n)$.

1.18. Kripke–Platek Set Theory, KP, is the theory in the language of $\in$ which is a proper subset of ZFC and consists of the following axioms: Foundation Schema, Extensionality, Pairing, Unions, Δ_0-Separation Schema (i.e. the universal closure of $\forall x \exists z \forall x [x \in z \leftrightarrow x \in y \wedge \varphi(x, y)]$ where φ is a Δ_0-formula), and Δ_0-Collection Schema (i.e. the universal closure of $\forall y \forall x \in y \exists z [\varphi(x, y, z) \rightarrow \exists w \forall x \in y \exists z \in w \varphi(x, y, z)]$).

1.9. PROPOSITION (Feferman–Kreisel [13]). *If T is a theory in the language of $\in$ such that $KP \subset T$, then formula φ is absolute relative to T if and only if φ is a Δ_1^T formula.*

1.20. PROPOSITION. *Let $M \models_L \varphi$ be the relation that holds between structures M for L and formulas φ of L that hold in M. This relation is Δ_1^{KP}, and hence absolute for KP.*

Proposition 1.20 also holds with the language and appropriate satisfaction relation for $L_{\omega_1\omega}$ and $L_{\infty\omega}$. See [2].

1.21. PROPOSITION (Montague–Levy Reflection Theorem). *Let $\langle W_\alpha \mid \alpha$ an ordinal$\rangle$ be a series of transitive sets such that: if $\alpha < \beta$ then $W_\alpha \subsetneq W_\beta$ and if α is a limit ordinal then $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$.* Let $\mathrm{W} = \bigcup_{\alpha \text{ ordinal}} W_\alpha$. *Let $\varphi(x)$ be any formula in the language of $\in$. Then* $\mathrm{ZF} \vdash \forall_\alpha \exists_{\beta>\alpha}$ *[β is a limit ordinal $\wedge \forall x \in W_\beta(\varphi(x)^{(W)} \leftrightarrow \varphi(x)^{(W_\beta)})$].*

1.22. PROPOSITION (Mostowski–Shepherdson Collapsing Lemma). *Let $\langle M, E\rangle$ be a model of Extensionality (i.e. $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$) such that E is well founded. Then there is a unique transitive set M' and a unique isomorphism from $\langle M, E\rangle$ onto $\langle M', \in\rangle$.*

1.23. DEFINITION (Gödel's Constructible Universe).
$L_0 = \emptyset$
$L_{\alpha+1} = \{x \mid x$ *is definable from* $L_\alpha\}$
$L_\beta = \bigcup_{\alpha<\beta} L_\alpha$ *if β is a limit ordinal.*
$L = \bigcup_{\alpha \text{ ordinal}} L_\alpha$.
A set is constructible if it belongs to L.

1.24. PROPOSITION (Gödel [15]). *For every model V_1 of* ZF *there is a structure V_2 such that $V_2 \subset_{\mathrm{end}} V_1$ and $V_2 \models \mathrm{ZFC} + V = L$.*

1.25. PROPOSITION (Shoenfield Absoluteness Lemma [47]). *Let $V \models$* ZFC. *Then the $L^{(V)}$-standard model of second order arithmetic satisfies the same Σ_2^1 sentences as the V-standard model of second order arithmetic.*

1.26. PROPOSITION (Jensen–Solovay [22]). *If* ZFC *is consistent then there are two models of* ZFC, *V_1 and V_2, with $V_1 \subset_{\mathrm{end}} V_2$, $V_2 \models$ "$L = V_1$" and $V_2 \models$ "Every constructible subset of ω is Δ_3^1."*

2. Forcing is not absolute; consequences

2.1. Let $\varphi(x_1,\cdots,x_n)$ be a formula in the language of $\in$. Recall that $\varphi(x_1,\cdots,x_n)$ is said to be *absolute for a theory* T if whenever V_1, V_2 are models of T with $V_1 \subset_{\text{end}} V_2$ and $a_1,\cdots,a_n$ elements of V_1 then $V_1 \models \varphi(a_1,\cdots,a_n)$ if and only if $V_2 \models \varphi(a_1,\cdots,a_n)$. $\varphi(x_1,\cdots,x_n)$ is said to be *absolute in Gödel's sense* or *absolute for* L if, whenever $a_1,\cdots,a_n$ are elements of L, $\varphi(a_1,\cdots,a_n)$ holds if and only if $\varphi(a_1,\cdots,a_n)^{(L)}$ holds.

Note that "$\varphi(x_1,\cdots,x_n)$ is absolute for ZFC" does not quite imply "$\varphi(x_1,\cdots,x_n)$ is absolute in Gödel's sense" since the former refers only to set-models. (Consider a model of ZFC + $V \neq L$ + ZFC is inconsistent. Then, in this model every formula is absolute for ZFC.) If "ZFC is consistent" is assumed then the absoluteness of $\varphi(x_1,\cdots,x_n)$ implies the consistency of the statement "$\varphi(x_1,\cdots,x_n)$ is absolute in Gödel's sense."

2.2. Definition. *SM* is the sentence in the language of $\in$ which, in every model of ZFC, asserts the existence of a standard model for ZFC.

Cons(ZFC) is a fixed natural sentence in the language of ZFC which asserts that ZFC is consistent.

If T is definable in ZFC, Cons(T) will be the statement asserting T is consistent.

2.3. Proposition (Cohen [9]). *Assume SM. Then there is a minimal standard model for* ZFC. *This model satisfies* $V = L$.

Proof. Well known. (A generalization is proved later. See 3.4.) □

The model in 2.3 is unique. Call it L_*.

2.4. Proposition (Cohen [9]). *In* L_*, *SM is false.*

Proof. If *SM* is true in L_*, since L_* is standard this contradicts the minimality of L_*. □

By the Lowenheim–Skolem Theorem, it is clear that $\|L_*\| = \aleph_0$.

2.5. Proposition. *Assume SM. Then* $(\|L_*\| = \aleph_0)^{(L)}$. □

An application of the reflection theorem 1.21 yields the following.

2.6. Proposition. *Assume SM. Then there is an ordinal* β *such that* $L_\beta \models \|L_*\| = \aleph_0$. □

2.7. DEFINITION. Let β_0 be the least ordinal satisfying the conclusion of 2.6. SM_2 is the sentence in the language of ZFC asserting that $\exists \alpha > \beta_0\, L_\alpha \models \mathrm{ZFC}$.

In a similar fashion, SM_3, SM_4, etc. could be defined. $SM_\infty : \forall_\beta \exists \alpha > \beta$ $L_\alpha \models \mathrm{ZFC}$.

2.8. THEOREM. *Assume SM and SM_2. Then the function T^F is not absolute relative to* ZFC.

PROOF. Let V be an L_α as in SM_2. Then in V it is true, by the definition of SM_2, that $L_* \models \mathrm{ZFC}$ and that $V \models \| L_* \| = \aleph_0$.

But this implies that the following sentence (*) is true in V:
(*) "there is a subset, A, of ω and a binary relation E on A such that the structure $\langle A, E\rangle$ encodes a standard model of ZFC."

The Mostowski–Shepherdson Collapsing Lemma (1.22) (which is a consequence of ZFC) allows (*) to be written as $\exists A \subset \omega \exists E \subset A \times A$ [E is well founded & $\forall \varphi$ [φ is an axiom of ZFC$\rightarrow \langle A, E\rangle \models \varphi$]].

(*) is true in V since $\| L_* \| = \aleph_0$ implies the existence of a one-to-one function mapping L_* into ω. This function may be used to determine E, so that $\langle A, E\rangle \cong L_*$. (Obviously A may be chosen as ω.)

Since (*) is really a sentence about subsets of ω, it follows that the sentence "There is an A such that A encodes a standard model of ZFC." is a sentence, φ, in second order number theory which is true in the natural model of V. In other words, $V \models (\langle N, P(N)\rangle \models \varphi)$.

(To verify that φ may be taken as a sentence of second order number theory, two facts should be noted. First, the set of true sentences of first order number theory, while not arithmetically definable is hyperarithmetic and thus is definable in second order arithmetic. (For a proof, see Moschovakis [33].) This means that the satisfaction predicate, $\langle A, E\rangle \models \psi$, is definable in second order number theory.

The second fact which should be noted is that the Mostowski–Shepherdson Collapsing Lemma (1.22) allows the fact that a structure is isomorphic to a standard one to be expressed in second order number theory.)

On the other hand, by Proposition 2.4, in L_* φ is false in the natural model of second order arithmetic. But by Proposition 1.13 if T is the set of true sentences of first order arithmetic, there is a one-one recursive function, f, from sentences of second order arithmetic to sentences of first order arithmetic such that $\psi^f \in T^F$ if and only if $\langle N, P(N)\rangle \models \psi$. Moreover, $(\sim \psi)^f = \sim(\psi^f)$.

Applying this to φ, $(\varphi^f \in T^F)^{(V)}$, while $(\sim \varphi^f \in T^F)^{(L_*)}$. Since $V \supset_{\mathrm{end}} L_*$, this completes the proof. $\square$

The above result would be the main result of this section, i.e. forcing is not absolute relative to ZFC, if it were not for the presence of assumptions SM and SM_2 in the hypothesis. It turns out that these assumptions can be replaced by Cons(ZFC) if the proper choice of a sentence for T^F is made.

The elimination of SM and SM_2 will require the consideration of non-standard models. As a tool towards this end, Lemma 2.10 is proved.

2.9. Let $\langle V, E\rangle$ be a model of ZFC, where it is not necessarily assumed that $\langle V, E\rangle$ is isomorphic to a standard model. Let K be a language with a finite number of relation and function symbols. (It is assumed here that K is defined so that it is set-theoretically well behaved, e.g. K is a subset of the hereditarily finite sets.) If follows that $K^{(V)}$ is essentially identical with K.

For definiteness, assume that the relation symbols of K are $R_1, \cdots, R_K$, where R_i is an r_i-ary relational symbol, and the function symbols of K are $f_1, \cdots, f_m$, where f_j is an s_j-ary functional symbol.

Now let $\mathcal{M} \in V$ be such that $\langle V, E\rangle \models \mathcal{M}$ is a structure for K. Then $\langle V, E\rangle \models \exists M \exists R_1 \cdots \exists R_k \exists f_1 \cdots \exists f_{m_{r_l}} \mathcal{M} = \langle M, R_1, \cdots, R_k, f_1, \cdots, f_m\rangle$. Let $M_E = \{x \in V \mid xEM\}$ and $R_{i_E} = \{\vec{a} \in V \mid \langle V, E\rangle \models R_i(a_1, \cdots, a_{r_i})\}$. Define f_{j_E} as a function from $M_E^{s_j}$ to M_E by $f_{j_E}(a_1, \cdots, a_{s_j}) = b$ if and only if $\langle V, E\rangle \models (\mathcal{M} \models f_i\,(a_1, \cdots, a_{s_j}) = b)$. Let $\mathcal{M}_E = \langle M_E, R_{1_E}, \cdots, R_{k_E}, \cdots, f_{m_E}\rangle$. $\mathcal{M}_E$ is clearly a K-structure.

2.10. Lemma. *Let $\langle V, E\rangle$, K, $\mathcal{M}$, and $\mathcal{M}_E$ be as above. Let φ be a sentence in $K(M_E)$. Then $\mathcal{M}_E \models \varphi$ if and only if $\langle V, E\rangle \models (\mathcal{M} \models \varphi)$.*

Proof. This is by induction on the complexity of φ.

i) If φ is atomic then $(\mathcal{M} \models \varphi)^{(V,E)}$ if and only if $\mathcal{M}_E \models \varphi$ by the definition of $\mathcal{M}_E$.

ii) If φ is $\psi \wedge \chi$ then $\mathcal{M}_E \models \varphi \Leftrightarrow \mathcal{M}_E \models \Psi$ and $\mathcal{M}_E \models \chi$

$\Leftrightarrow$ (by induction) $(\mathcal{M} \models \Psi)^{(V,E)}$ & $(\mathcal{M} \models \chi)^{(V,E)}$

$\Leftrightarrow$ $(\mathcal{M} \models \varphi)^{(V,E)}$.

iii) If φ is $\sim \psi$ then $\mathcal{M}_E \models \varphi \Leftrightarrow$ not $\mathcal{M}_E \models \psi$

$\Leftrightarrow$ (by induction) not $(\mathcal{M} \models \psi)^{(V,E)}$

$\Leftrightarrow$ $(\mathcal{M} \models \varphi)^{(V,E)}$.

iv) If φ is $\exists x \psi(x)$ then $\mathcal{M}_E \models \varphi \Leftrightarrow \exists a \in \mathcal{M}_E\ \mathcal{M}_E \models \psi(a)$

$\Leftrightarrow$ $\exists a \in V$ $(aEM$ and $\mathcal{M}_E \models \psi(a))$

$\Leftrightarrow$ (by induction) $\exists a \in V$ $(aEM$ and $(\mathcal{M} \models \psi(a))^{(V,E)}$

$\Leftrightarrow$ $(\exists a \in M$ and $\mathcal{M} \models \psi(a))^{(V,E)}$

$\Leftrightarrow$ $(\mathcal{M} \models \exists x \psi(x))^{(V,E)}$

$\Leftrightarrow$ $(\mathcal{M} \models \varphi)^{(V,E)}$. $\square$

In [1], J. W. Addison explicitly exhibited (in ZFC) a formula $\mathrm{Constr}(X)$ of second order number theory such that $\psi = \forall X\, \mathrm{Constr}(X)$ is true in the natural model $(\langle N, P(N)\rangle)$ if and only if every subset of the integers is constructible (i.e. $P(N) \subset L$). Let f be the transformation of Proposition 1.11. Then ψ^f will be the sentence that illustrates the non-absoluteness of the function $T \to T^F$.

2.11. THEOREM. *The consistency of* ZFC + *"There is a non-constructible subset of the integers" implies the non-absoluteness of the function T^F relative to* ZFC.

PROOF. As in the proof of Theorem 2.8, it suffices to show that there are two models, V_1 and V_2, with $V_2 \subset_{\mathrm{end}} V_1$ and a sentence, ψ, in the language of second order number theory such that $(\langle N, P(N)\rangle \models \psi)^{(V_2)}$ while $(\langle N, P(N)\rangle \models \sim \psi)^{(V_1)}$.

The choice of ψ, as mentioned above, is $\forall X\, \mathrm{Constr}(X)$. V_1 and V_2 are chosen as follows. By hypothesis, there is a model of ZFC + "There is a non-constructible subset of the integers." Let V_1 be such a model. It is clear that $(\langle N, P(N)\rangle \models > \psi)^{(V_1)}$.

By Gödel's constructibility result, every model of ZFC has an inner model, its constructible universe. Let V_2 be this inner model for V_1, i.e. $V_2 = L^{(V_1)}$. Since V_2 is an inner model, $V_1 \supset_{\mathrm{end}} V_2$. Moreover, since $V_2 \models V = L$, it follows that $(\langle N, P(N)\rangle \models \psi)^{(V_2)}$ is true. This completes the proof. □

2.12. REMARK. Although $\mathrm{Constr}(X)$ is the same sentence as Addison's, $V_2 \models (\langle N, P(N)\rangle \models \forall X\, \mathrm{Constr}(X))$ does not imply that $\langle N_E, P(N)_E\rangle$ is constructible. Lemma 2.10 does imply $\langle N_E, P(N)_E\rangle \models \forall X\, \mathrm{Constr}(X)$ but non-well ordered constructing sequences may have been used.

2.13. COROLLARY. $\mathrm{Cons}(\mathrm{ZFC} + V = L) \to T^F$ *is not absolute relative to* ZFC.

PROOF. By a Cohen-forcing argument with *SM* eliminated (see [9, pp. 125, 147] for details), $\mathrm{Cons}(\mathrm{ZFC} + V = L)$ implies Cons(ZFC + "There is a non-constructible subset of the integers.") □

2.14. COROLLARY. $\mathrm{Cons}(\mathrm{ZFC}) \to T^F$ *is not absolute relative to* ZFC.

PROOF. Gödel's constructibility result [15] is

$$\mathrm{Cons}(\mathrm{ZFC}) \to \mathrm{Cons}(\mathrm{ZFC} + V = L). \qquad \square$$

In the proof of Theorem 2.11, $V_2 = L^{(V_1)}$. This observation proves the following result.

2.15. THEOREM. *If* ZFC *is consistent, then it is consistent that* T^F *is not absolute in Gödel's sense.* □

This answers a question of G. Sacks.

The above is also the best possible result in the present meta-theory, since Gödel's constructibility result may be stated as Cons(ZFC)→Cons(ZFC+ "Everything is absolute in Gödel's sense").

Each of the following corollaries also has its analogue in terms of absoluteness in Gödel's sense.

2.16. COROLLARY. *If* ZFC *is consistent then the ternary forcing relation* $M \Vdash_T \varphi$, *is not absolute for* ZFC.

PROOF. Definition 1.10 states that $T^F = \{\varphi \mid M \Vdash_T \sim \sim \varphi$ for all $M \in \Sigma(T)\}$. Since satisfaction is absolute for ZFC (Proposition 1.20), so is $\Sigma(T)$. □

2.17. REMARK. For a particular T, it is possible for the binary relation $M \Vdash_T \varphi$ to be absolute for ZFC. An example occurs when T is the theory of fields.

2.18. COROLLARY. *If* ZFC *is consistent then the notion of* T*-genericity is not absolute for* ZFC.

PROOF. Satisfaction is absolute (1.20) and any generic model must satisfy T^F. □

2.19. COROLLARY. *If* ZFC *is consistent, then it is consistent that the generic division rings are a disjoint class from the (generic division rings)*$^{(L)}$.

PROOF. Use Proposition 1.12. □

The claim [40] that generic structures are the proper realization of the notion of algebraic closedness is perhaps rendered more dubious by this last result. However, recent results of Shelah [46] and Osofsky [34] illustrate that algebraic notions themselves may fail to be absolute for ZFC. In general, a problem may arise when inclusions or injections must be taken into account. But this is just what is important for considerations of algebraic closedness. Viewed in this light, Corollary 2.19 is not as surprising.

2.20. COROLLARY. *There is a sentence* φ *in the language of division rings such that a generic may satisfy* φ *only if the Continuum Hypothesis holds. It is also consistent with* ZFC *that* φ *is true in every generic division ring.*

PROOF. Let φ be the recursive translation of $\forall X\, \mathrm{Constr}(X)$ into the language of division rings (as given by 1.12). □

2.21. THEOREM. *If* ZFC *is consistent then it is consistent with* ZFC *that the infinite forcing companion of a finitely axiomatized theory not be constructible.*

PROOF. Let the theory be the axioms for group theory, G. The model of ZFC to be used is the one given by the Jensen-Solovay model (1.26). This model, V_2, satisfies "every constructible subset of ω is analytical". By 1.11 G^F is recursively equivalent to full second order arithmetic and hence is certainly not analytical. Hence $(G^F)^{(V_2)} \notin L^{(V_2)}$. □

This last result is a further answer to an innocent question of Martin Davis at the 1970 International Congress at Nice. Davis asked if N^F was arithmetical.

2.22. REMARK. If there is a standard model for ZFC (i.e. SM holds) then the proofs of 2.11–2.20 yield more information. The counterexamples to the absoluteness of T^F, etc. may be taken to be standard models.

Even without assuming SM stronger results than have been stated were shown.

2.23. DEFINITION. Let $V_1, V_2 \models$ ZFC. V_1 is an *ordinal end extension* of V_2, $V_1 \supset_{\mathrm{ord}} V_2$, if $V_1 \supset_{\mathrm{end}} V_2$ and the ordinals of V_1 are all in V_2.

$\varphi(\vec{X})$ is said to be *ordinally absolute for* ZFC if whenever $V_1 \supset_{\mathrm{ord}} V_2$ and $\vec{a} \in V_2$ then $V_2 \models \varphi(\vec{a}) \Leftrightarrow V_1 \models \varphi(\vec{a})$.

The next two lemmas are easy. See [21].

2.24. LEMMA. *Let* $V = \mathrm{ZFC}$. *Then* $L^{(V)} \subset_{\mathrm{ord}} V$. □

2.25. LEMMA. *If* V_1 *is a Cohen extension of* V_2 *then* $V_1 \supset_{\mathrm{ord}} V_2$. □

The models used in the proofs of non-absoluteness of T^F etc. were all of the above two types. Thus the following has already been shown.

2.26. COROLLARY. *If* ZFC *is consistent then* T^F *is not ordinally absolute relative to* ZFC.

Robinson's questions on resultants

2.27. Let T be an inductive theory in some language K. Let $\varphi(x_1, \cdots, x_n)$ be a formula in K with free variables $x_1, \cdots, x_n$. A. Robinson proved [37, 7.1] that there is a set of existential formulas, $\{\varphi_{\mu\lambda}(x_1, \cdots, x_n) \mid \mu \in I, \lambda \in J\}$ (I and J are index sets), such that if M is a model of T and $a_1, \cdots, a_n$ are elements of M, then $M \Vdash \sim \sim \varphi(a_1, \cdots, a_n)$ if and only if there is a $\mu \in I$ such that for all $\lambda \in J$ $M \models \varphi_{\mu\nu}(a_1, \cdots, a_n)$. In the terminology of infinitary logic

$$M \Vdash \sim\sim \varphi(a_1, \cdots, a_n) \Leftrightarrow M \models \bigvee_{\mu \in I} \bigwedge_{\lambda \in J} \varphi_{\mu\lambda}(a_1, \cdots, a_n).$$

However, Robinson's proof of this theorem is very non-effective and thus gives no information as to how the $\varphi_{\mu\lambda}$ may be found.

Of course, for a particular T, there may be a clear recursive method for finding such resultants. For example, if T is the theory of ordered fields, then the algorithm of Tarski and Seidenberg [20, VI.7–VI.9] for real closed fields will suffice.

Robinson raised the natural question: Can the $\varphi_{\mu\lambda}$ be chosen by some algorithmic method in the general case?

The first problem encountered in any attempt to answer such a question is deciding just what should tbe meant by "algorithmic". The most common interpretation is via Church's Thesis, i.e. "algorithmic" means "recursive". Under this interpretation Robinson's question may be answered in the negative. However, this is for trivial reasons, e.g. I and J may be required to be uncountable sets.

To avoid such trivialities, other interpretations of algorithms have been suggested as appropriate for model theory. Two such examples are recursive functionals [42] and Robinson's recent notion of "sets of derivations" [41].

One possible criterion that seems practically self-evident is that a notion of algorithm must be absolute for ZFC. However, to avoid cardinality problems only a weaker criterion will be assumed.

2.28. CRITERION. A notion of algorithm must be ordinally absolute for ZFC.

This criterion is sensible since an algorithm should not depend on which universe of set theory the decisions are made in, at least when the universes are not radically different "sizes". In any case, all of the examples cited above satisfy this criterion.

2.29. THEOREM. *Assume Criterion* 2.28. *There are theories T and formulas $\varphi(x_1, \cdots, x_n)$ in the language of T such that there is no algorithm for finding the resultants for $\varphi(x_1, \cdots, x_n)$.*

PROOF. By Robinson's result, if $M \models T$ then $M \Vdash \sim\sim \varphi(a_1, \cdots, a_n)$ if and only if $M \models \bigvee_\mu \bigwedge_\lambda \varphi_{\mu\lambda}(a_1, \cdots, a_n)$. But by the absoluteness of satisfaction (following 1.20) the right side of this equivalence is absolute for ZFC. Thus if the $\varphi_{\mu\lambda}$ were chosen by a method which was ordinally absolute for ZFC, the left side would also have to be ordinally absolute for ZFC. Since this would contradict

Corollary 2.26 it follows that the $\varphi_{\mu\lambda}$ may not be so chosen. Hence by Criterion 2.28 the $\varphi_{\mu\lambda}$ may not be chosen algorithmically. □

Non-axiomatizability of generics

2.30. Angus Macintyre proved in [29] that for certain theories T, the class of T-generic structures is not axiomatizable by a single sentence of $L_{\omega_1\omega}$. (For a description of $L_{\omega_1\omega}$ see [23]).

This result follows (modulo the consistency of ZFC) easily from the previous results in this paper and a general observation of K. Jon Barwise [2]. (In the following KP are the Kripke–Platek axioms. It suffices for the present purposes to note that KP $\subset$ ZFC.)

2.31. FACT (theorem 3.5 of [2]). Let L be a language. A class H of L-structures that is closed under isomorphism is axiomatizable by a single sentence of $L_{\omega_1\omega}$ if and only if there is a predicate $P(x, y)$ that is absolute for KP (and hence for ZFC) and a hereditarily countable set, a, such that $H = \{M \mid P(M, a)\}$.

The set mentioned in Fact 2.31 is essentially the transitive closure of the language. Since this will not affect Corollary 2.18 the following is immediate.

2.32. THEOREM. *If* ZFC *is consistent then there are theories T such that the class of T-generic structures are not axiomatizable by a single sentence of $L_{\omega_1\omega}$.*

The generic hierarchies

2.33. Several people (G. Cherlin [8] and D. Saracino [44], H. Simmons [48], and later J. Hirschfeld and W. Wheeler [19]) have shown how the class of generic structures of a theory T may be obtained as a limit of a hierarchy of classes of structures.

These hierarchies are all defined in the following fashion:

1) E_1 is the class of existentially complete structures.

2) E_n is defined in terms of E_{n-1}.

In each case, $E_n \subset E_{n-1}$ and $G = \bigcap_{n\in\omega} E_n$. ($G$ is the class of generic structures.)

As was noted in the introduction, if T is a theory absolute for ZFC then E_1 is a class that is absolute relative to ZFC. (This may be proved by noting that E_1 may be axiomatized by a single sentence of $L_{\omega_1\omega}$ [50, 2.2] and applying Fact 2.28.) On the other hand, Corollary 2.18 has established that G is not an absolute class.

Since both of these results are theorems of ZFC it follows that

(*) ZFC $\vdash$ "If ZFC is consistent then there is an integer n such that E_n is not absolute for ZFC (for a general theory T) while E_{n-1} is absolute for ZFC (for any theory T)."

Of course, the value of n in (*) will depend on the particular hierarchy chosen. Moreover, it is not clear, *a priori*, that n will not depend on the model of ZFC in which it is calculated.

However, if the hierarchy is taken to be the Hirschfeld hierarchy, then Hirschfeld's work [17] makes the calculation of n straightforward. It turns out that $n = 4$. A similar calculation can be done for the other hierarchies.

2.34. Definition. E_1 is the class of existentially complete models of arithmetic.

$M \in E_{n+1}$ if and only if $M \in E_n$ and every Σ_{n+1} formula in the language of M that holds in some extension of M in E_n holds in M. (Recall that a formula of first order theory is Σ_n if it is a formula with only n alternations of unbounded quantifiers. See [17].)

2.35. Definition (Hirschfeld). Let $M \in E_1$. $\langle N, S_M \rangle$ is a structure for the language of second order arithmetic, where N is ω (which is always definable in M) and S_M is the set of subsets of N that are existentially definable in M. Note that this definition is absolute for ZFC.

2.36. Definition. A sentence of second order number theory is Σ_n^1 if it has at most $n-1$ alternations of set quantifiers beginning with an existential one.

2.37. Definition. An ω-structure $\langle N, S \rangle$, where $S \subset P(N)$, for second-order number theory is a *β_n-model* if any Σ_n^1 formula with parameters from $\langle N, S \rangle$ holds in $\langle N, P(N) \rangle$ (the standard model) if and only if it holds in $\langle N, S \rangle$.

Note: The notion of β_n-model is not absolute for ZFC for arbitrary n.

2.38. Proposition (Hirschfeld [17]). *M is a β_n model if and only if $M \in E_{n+1}$.* $\square$

2.39. Theorem. *Let T be the theory of first order number theory and let E_n be as in Definition 2.34 for T. Then membership in E_3 is absolute for* ZFC *while membership in E_4 is not absolute for* ZFC.

Proof. The Shoenfield Absoluteness Lemma (1.25) essentially states that the class of β_2-models is absolute for ZFC. It follows by Proposition 2.38 that E_3 is an absolute class.

On the other hand Addison [1] showed that the sentence $\forall X\, \mathrm{Constr}(X)$ is π^1_3, and hence (e.g. as in the proof of Theorem 2.11) the class of β_3-models is not absolute for ZFC. By Proposition 2.38 it follows that E_4 is not an absolute class.

3. ZFC is not important

The previous section presented a proof that genericity and related concepts are not absolute relative to ZFC. To what extent was this a consequence of the strength of ZFC? Put another way, is it possible that there is an extension of ZFC, ZFC $+\Phi$, such that T^F is absolute relative to ZFC$+\Phi$? This situation is not ruled out by the results of the previous section (*a fortiori*, forcing is not absolute relative to any weaker set theory, e.g. KP or ZF).

If the strength of ZFC was significant then it would be natural to look for new axioms Φ such that forcing is absolute relative to ZFC$+\Phi$. Of course, these axioms should be "true" ones, i.e. they should hold in the universe. This turns out not to be a viable idea.

3.1. Definition. Let Φ be any set of sentences in the language of ZFC such that Φ is definable in ZFC. SM_Φ is the sentence asserting that ZFC$+\Phi$ has a standard model.

In this section Φ will always be a definable set of sentences. Thus SM_Φ will always be defined.

The aim of this section is to prove the following.

3.2. Theorem (ZFC). *If* ZFC$+\Phi+SM_\Phi$ *is consistent then* T^F *is not absolute relative to* ZFC$+\Phi$.

It is unknown if Con (ZFC$+\Phi$) implies the non-absoluteness of T^F relative to ZFC$+\Phi$, or even if SM_Φ is sufficient. However, the following holds.

3.3. Theorem (ZFC). *If* ZFC$+\Phi+SM_\Phi$ *has a standard model then* T^F *is not absolute relative to* ZFC$+\Phi$.

While Theorem 3.3 is a direct consequence of Theorem 3.2, 3.3 will nonetheless be proved first since its proof will serve as a model for that of 3.2.

3.4. Lemma (ZFC). SM_Φ *implies the existence of a countable standard model of* ZFC$+\Phi+\sim SM_\Phi$.

Proof. By hypothesis there is a standard model, V_1, of ZFC$+\Phi$. Assume for contradiction that every such model satisfies SM_Φ. In particular V_1 satisfies SM_Φ.

Hence there is a $V_2 \in V_1$ such that $V_1 \models$ "V_2 is a standard model of ZFC$+\Phi$."

Since V_1 is standard, it follows that V_2 is as well, and, by the absoluteness of satisfaction, $V_2 \models \mathrm{ZFC} + \Phi$.

By assumption $V_2 \models SM_\Phi$ as well. This allows the argument to be repeated, and in this fashion an $\in$-chain of models may be obtained: $V_1 \ni V_2 \ni V_3 \ni \cdots V_n \ni V_{n+1} \ni \cdots$. But the existence of such a chain contradicts the axiom of regularity. (Consider $\bigcup_n V_n$.)

Thus there is a standard model of $\mathrm{ZFC} + \Phi + \sim SM_\Phi$. A countable such model exists by the Löwenheim–Skolwm Theorem. □

3.5. Definition. V is a *minimal model* for $\mathrm{ZFC} + \Phi$ if V is countable, standard, transitive and $V \models \mathrm{ZFC} + \Phi + \sim SM_\Phi$.

3.6. Corollary (ZFC). *SM_Φ implies the existence of a minimal standard model of* $\mathrm{ZFC} + \Phi$. □

3.7. Definition. $ST_\Phi(\langle V, E\rangle)$ is the sentence of second order arithmetic that asserts that $\langle V, E\rangle$ is isomorphic to a standard transitive model of $\mathrm{ZFC} + \Phi$. (More precisely:

"E is a binary relation on V and $\forall m \in V \forall n \in V \forall k \in V \{[mEn \leftrightarrow mEk] \rightarrow n = k] \;\&\; \forall S \subset V \exists x \in S \forall y \in S(\sim yEx) \;\&\; \langle V, E\rangle$ is a model of $\mathrm{ZFC} + \Phi$."

The Mostowski–Shepherdson Collapsing Lemma, 1.22, has been used implicitly.)

3.8. Proof of Theorem 3.3. By hypothesis there is a standard model, V_1, of $\mathrm{ZFC} + \Phi + SM_\Phi$. Since $V_1 \models SM_\Phi$ there is a V_1-standard model, V_2, such that $V_2 \models \mathrm{ZFC} + \Phi$. Moreover, by Lemma 3.4, relativized to V_1, V_2 may be chosen to be minimal, i.e. such that $(V_2 \models \sim SM_\Phi)^{(V_1)}$.

By the definition of minimality, V_2 is V_1-transitive and V_1-standard. Hence $V_1 \supset_{\mathrm{end}} V_2$. Moreover, V_2 is actually standard (since V_1 is) and $V_2 \models \sim SM_\Phi$. Since $V_2 \models \sim SM_\Phi$, $(\langle \omega, P(\omega)\rangle \models \sim \exists X\, ST_\Phi(X))^{(V_2)}$.

On the other hand, $V_1 \models SM_\Phi$ and therefore $(\langle \omega, P(\omega)\rangle \models \exists X\, ST_\Phi(X))^{(V_1)}$. Hence the theory of second order arithmetic in V_1 differs from that in V_2. Proposition 1.11 implies that T^F is not the same in V_1 and V_2 when T is the theory of groups. Thus T^F is not absolute relative to $\mathrm{ZFC} + \Phi$. □

3.9. Proof of Theorem 3.2. By hypothesis, there is a model, $\langle V, E\rangle$, not necessarily standard, of $\mathrm{ZFC} + \Phi + SM_\Phi$. Since $\langle V, E\rangle \models SM_\Phi$ there is a V-standard model, M, of $\mathrm{ZFC} + \Phi$ in V. By Lemma 3.4 (relativized to $\langle V, E\rangle$) M may be chosen such that $\langle V, E\rangle \models$ "M is a minimal model of $\mathrm{ZFC} + \Phi$." But then as in the proof of Theorem 3.3,

$$[M \models (\langle \omega, P(\omega) \rangle \models \sim \exists X ST_\Phi(X))]^{(V,E)}$$

while $\langle V, E \rangle \models (\langle \omega, P(\omega) \rangle \models \exists X ST_\Phi(X))$.

If $\langle V, E \rangle \supset_{\text{end}} M$ the proof would be completed. However M need not even be a model although it is a V-model. The tool to overcome this difficulty is precisely Lemma 2.10. $\langle V, E \rangle \supset_{\text{end}} M_E$ and by the conclusion of the lemma, $M_E \models (\langle \omega, P(\omega) \rangle \models \sim \exists X ST_\Phi(X))$. □

Corollaries, in parallel with those of 2.15–2.19, follow directly. Since these are obvious they are omitted. However the following is more interesting.

3.10. Corollary. *If* ZFC *is consistent then* T^F *is not absolute relative to* ZFC + *Martin's Axiom.*

3.11. Corollary. *If* ZFC *is consistent then* T^F *is not absolute relative to* ZFC + $V = L$.

3.12. Corollary. *If it is consistent that* ZFC + *"there is a measureable cardinal" has a standard model, then* T^F *is not absolute relative to* ZFC + *"there is a measureable cardinal".*

The same type of corollary holds for any proposed axiom. In particular, it holds for all large cardinal axioms.

3.13. It is not known whether the assumption of the consistency of a standard model of ZFC + Φ in 3.2, or the assumption SM_{SM_Φ} in 3.3 may be reduced to SM_Φ. The method of proof of these theorems will not suffice in this case since ZFC + Φ + $SM_\Phi \not\Rightarrow$ Cons (ZFC + Φ + SM_Φ).

However, the hypothesis of 3.2 is reasonable for any "true" set theory. Expressed another way, 3.2 says that any set theory, S, that extends ZFC, and relative to which forcing is absolute, is not consistent with the existence of a standard model of S. But as Cohen [9, p. 79] points out, such a consistency statement about a "true" set theory is certainly "true".

3.14. As in section 2 all of the above results relativize to the consistency of non-absoluteness in Gödel's sense.

3.15. Considerations of absoluteness in Gödel's sense lead to the following natural question. (It was raised by M. Boffa and A. Macintyre.) Suppose it is assumed that for all countable, inductive, consistent and constructible theories T, $(T^F)^{(L)} = T^F$. How strong is this assumption?

It turns out that this is actually equivalent to $P(\omega) = P(\omega)^{(L)}$. In the following theorem it is necessary to recall the definition of Constr(X), that appears just prior to 2.11.

3.16. THEOREM (ZFC). $\forall T$ (*T is countable, inductive and consistent theory*) *and if* $T \in L$ *then* $T^F = (T^F)^{(L)} \Leftrightarrow \langle N, P(N)\rangle \models \forall X\, \mathrm{Constr}(X)$.

PROOF. ($\Rightarrow$) *Let V* be an arbitrary model of ZFC such that for all countable, consistent, inductive and constructible T, $T^F = (T^F)^{(L)}$. By the definition of Constr(X), it is obvious that

$$(\langle N, P(N)\rangle \models \forall X\, \mathrm{Constr}(X))^{(L^{(V)})}.$$

Now let K be the axioms for group theory. By 1.11, "K^F is recursively equivalent to second order number theory" is true in both V and $L^{(V)}$.

Since $K^F = (K^F)^{(L)}$, it follows that second order number theory in $L^{(V)}$ must be the same as second order number theory in V. Thus

$$(\langle N, P(N)\rangle \models \forall X\, \mathrm{Constr}(X))^{(V)}.$$

($\Leftarrow$) This direction will only be sketched. The essential idea is that since the constructible subsets of ω coincide with all of the subsets of ω, model theory is essentially the same in V and L. More precisely, applications of the Lowenheim–Skolem Theorem allow the duplication of any tree of countable structures inside L. Given 1.6, this is sufficient to prove $T^F = (T^F)^{(L)}$. □

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Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel